

Probabilistic Analysis of Power Assignments

Maurits de Graaf^{1,2} and Bodo Manthey¹

¹University of Twente, Department of Applied Mathematics, Enschede, Netherlands
 m.degraaf/b.manthey@utwente.nl

²Thales Nederland B. V., Huizen, Netherlands

March 25, 2014

A fundamental problem for wireless ad hoc networks is the assignment of suitable transmission powers to the wireless devices such that the resulting communication graph is connected. The goal is to minimize the total transmit power in order to maximize the life-time of the network. Our aim is a probabilistic analysis of this power assignment problem. We prove complete convergence for arbitrary combinations of the dimension d and the distance-power gradient p . Furthermore, we prove that the expected approximation ratio of the simple spanning tree heuristic is strictly less than its worst-case ratio of 2.

Our main technical novelties are two-fold: First, we find a way to deal with the unbounded degree that the communication network induced by the optimal power assignment can have. Minimum spanning trees and traveling salesman tours, for which strong concentration results are known in Euclidean space, have bounded degree, which is heavily exploited in their analysis. Second, we apply a recent generalization of Azuma-Hoeffding's inequality to prove complete convergence for the case $p \geq d$ for both power assignments and minimum spanning trees (MSTs). As far as we are aware, complete convergence for $p > d$ has not been proved yet for any Euclidean functional.

1 Introduction

Wireless ad hoc networks have received significant attention due to their many applications in, for instance, environmental monitoring or emergency disaster relief, where wiring is difficult. Unlike wired networks, wireless ad hoc networks lack a backbone infrastructure. Communication takes place either through single-hop transmission or by relaying through intermediate nodes. We consider the case that each node can adjust its transmit power for the purpose of power conservation. In the assignment of transmit powers, two conflicting effects have to be taken into account: if the transmit powers are too low, the resulting network may be disconnected. If the transmit powers are too high, the nodes run out of energy quickly. The goal of the power assignment problem is to assign transmit powers to the transceivers such that the resulting network is connected and the sum of transmit powers is minimized [13].

1.1 Problem Statement and Previous Results

We consider a set of vertices $X \subseteq [0, 1]^d$, which represent the sensors, $|X| = n$, and assume that $\|u - v\|^p$, for some $p \in \mathbb{R}$ (called the *distance-power gradient* or *path loss exponent*), is the power required to successfully transmit a signal from u to v . This is called the power-attenuation model, where the strength of the signal decreases with $1/r^p$ for distance r , and is a simple yet very common model for power assignments in wireless networks [19]. In practice, we typically have $1 \leq p \leq 6$ [16].

A power assignment $\mathbf{pa} : X \rightarrow [0, \infty)$ is an assignment of transmit powers to the nodes in X . Given \mathbf{pa} , we have an edge between two nodes u and v if both $\mathbf{pa}(u), \mathbf{pa}(v) \geq \|u - v\|^p$. If the resulting graph is connected, we call it a *PA graph*. Our goal is to find a PA graph and a corresponding power assignment \mathbf{pa} that minimizes $\sum_{v \in X} \mathbf{pa}(v)$. Note that any PA graph $G = (X, E)$ induces a power assignment by $\mathbf{pa}(v) = \max_{u \in X: \{u, v\} \in E} \|u - v\|^p$.

PA graphs can in many aspects be regarded as a tree as we are only interested in connect- edness, but it can contain more edges in general. However, we can simply ignore edges and restrict ourselves to a spanning tree of the PA graph.

The minimal connected power assignment problem is NP-hard for $d \geq 2$ and APX-hard for $d \geq 3$ [3]. For $d = 1$, i.e., when the sensors are located on a line, the problem can be solved by dynamic programming [10]. A simple approximation algorithm for minimum power assignments is the minimum spanning tree heuristic (MST heuristic), which achieves a tight worst-case approximation ratio of 2 [10]. This has been improved by Althaus et al. [1], who devised an approximation algorithm that achieves an approximation ratio of 5/3. A first average-case analysis of the MST heuristic was presented by de Graaf et al. [4]: First, they analyzed the expected approximation ratio of the MST heuristic for the (non-geometric, non-metric) case of independent edge lengths. Second, they proved convergence of the total power consumption of the assignment computed by the MST heuristic for the special case of $p = d$, but not of the optimal power assignment. They left as open problems, first, an average-case analysis of the MST heuristic for random geometric instances and, second, the convergence of the value of the optimal power assignment.

Other power assignment problems studied include the k -station network coverage problem of Funke et al. [5], where transmit powers are assigned to at most k stations such that X can be reached from at least one sender, or power assignments in the SINR model [7, 9].

1.2 Our Contribution

In this paper, we conduct an average-case analysis of the optimal power assignment problem for Euclidean instances. The points are drawn independently and uniformly from the d -dimensional unit cube $[0, 1]^d$. We believe that probabilistic analysis is a better-suited measure for performance evaluation in wireless ad hoc networks, as the positions of the sensors – in particular if deployed in areas that are difficult to access – are naturally random.

Roughly speaking, our contributions are as follows:

1. We show that the power assignment functional has sufficiently nice properties in order to apply Yukich's general framework for Euclidean functionals [25] to obtain concentration results (Section 3).
2. Combining these insights with a recent generalization of the Azuma-Hoeffding bound [24], we obtain concentration of measure and complete convergence for all combinations of d and $p \geq 1$, even for the case $p \geq d$ (Section 4). In addition, we obtain complete

convergence for $p \geq d$ for minimum-weight spanning trees. As far as we are aware, complete convergence for $p \geq d$ has not been proved yet for such functionals. The only exception we are aware of are minimum spanning trees for the case $p = d$ [25, Sect. 6.4].

3. We provide a probabilistic analysis of the MST heuristic for the geometric case. We show that its expected approximation ratio is strictly smaller than its worst-case approximation ratio of 2 [10] for any d and p (Section 5).

Our main technical contributions are two-fold: First, we introduce a transmit power redistribution argument to deal with the unbounded degree that graphs induced by the optimal transmit power assignment can have. The unboundedness of the degree makes the analysis of the power assignment functional PA challenging. The reason is that removing a vertex can cause the graph to fall into a large number of components and it might be costly to connect these components without the removed vertex. In contrast, the degree of any minimum spanning tree, for which strong concentration results are known in Euclidean space for $p \leq d$, is bounded for every fixed d , and this is heavily exploited in the analysis. (The concentration result by de Graaf et al. [4] for the power assignment obtained from the MST heuristic also exploits that MSTs have bounded degree.)

Second, we apply a recent generalization of Azuma-Hoeffding's inequality by Warnke [24] to prove complete convergence for the case $p \geq d$ for both power assignments and minimum spanning trees. We introduce the notion of *typically smooth* Euclidean functionals, prove convergence of such functionals, and show that minimum spanning trees and power assignments are typically smooth. In this sense, our proof of complete convergence provides an alternative and generic way to prove complete convergence, whereas Yukich's proof for minimum spanning trees is tailored to the case $p = d$. In order to prove complete convergence with our approach, one only needs to prove convergence in mean, which is often much simpler than complete convergence, and typically smoothness. Thus, we provide a simple method to prove complete convergence of Euclidean functionals along the lines of Yukich's result that, in the presence of concentration of measure, convergence in mean implies complete convergence [25, Cor. 6.4].

2 Definitions and Notation

Throughout the paper, d (the dimension) and p (the distance-power gradient) are fixed constants. For three points x, y, v , we by \overline{xv} the line through x and v , and we denote by $\angle(x, v, y)$ the angle between \overline{xv} and \overline{yv} .

A *Euclidean functional* is a function F^p for $p > 0$ that maps finite sets of points in $[0, 1]^d$ to some non-negative real number and is translation invariant and homogeneous of order p [25, page 18]. From now on, we omit the superscript p of Euclidean functionals, as p is always fixed and clear from the context.

PA_B is the canonical boundary functional of PA (we refer to Yukich [25] for boundary functionals of other optimization problems): given a hyperrectangle $R \subseteq \mathbb{R}^d$ with $X \subseteq R$, this means that a solution is an assignment $\text{pa}(x)$ of power to the nodes $x \in X$ such that

- x and y are connected if $\text{pa}(x), \text{pa}(y) \geq \|x - y\|^p$,
- x is connected to the boundary of R if the distance of x to the boundary of R is at most $\text{pa}(x)^{1/p}$, and
- the resulting graph, called a *boundary PA graph*, is either connected or consists of connected components that are all connected to the boundary.

Then $\text{PA}_B(X, R)$ is the minimum value for $\sum_{x \in X} \text{pa}(x)$ that can be achieved by a boundary PA graph. Note that in the boundary functional, no power is assigned to the boundary. It is straight-forward to see that PA and PA_B are Euclidean functionals for all $p > 0$ according to Yukich [25, page 18].

For a hyperrectangle $R \subseteq \mathbb{R}^d$, let $\text{diam } R = \max_{x, y \in R} \|x - y\|$ denote the diameter of R . For a Euclidean functional F , let $F(n) = F(\{U_1, \dots, U_n\})$, where U_1, \dots, U_n are drawn uniformly and independently from $[0, 1]^d$. Let

$$\gamma_F^{d,p} = \lim_{n \rightarrow \infty} \frac{\mathbb{E}(F(n))}{n^{\frac{d-p}{d}}}.$$

(In principle, $\gamma_F^{d,p}$ need not exist, but it does exist for all functionals considered in this paper.)

A sequence $(R_n)_{n \in \mathbb{N}}$ of random variables *converges in mean* to a constant γ if $\lim_{n \rightarrow \infty} \mathbb{E}(|R_n - \gamma|) = 0$. The sequence $(R_n)_{n \in \mathbb{N}}$ *converges completely* to a constant γ if we have

$$\sum_{n=1}^{\infty} \mathbb{P}(|R_n - \gamma| > \varepsilon) < \infty$$

for all $\varepsilon > 0$.

Besides PA, we consider two other Euclidean functions: $\text{MST}(X)$ denotes the length of the minimum spanning tree with lengths raised to the power p . $\text{PT}(X)$ denotes the total power consumption of the assignment obtained from the MST heuristic, again with lengths raised to the power p . The MST heuristic proceeds as follows: First, we compute a minimum spanning tree of X . The let $\text{pa}(x) = \max\{\|x - y\|^p \mid \{x, y\} \text{ is an edge of the MST}\}$. By construction and a simple analysis, we have $\text{MST}(X) \leq \text{PA}(X) \leq \text{PT}(X) \leq 2 \cdot \text{MST}(X)$ [10].

For $n \in \mathbb{N}$, let $[n] = \{1, \dots, n\}$.

3 Properties of the Power Assignment Functional

After showing that optimal PA graphs can have unbounded degree and providing a lemma that helps solving this problem, we show that the power assignment functional fits into Yukich's framework for Euclidean functionals [25].

3.1 Degrees and Cones

As opposed to minimum spanning trees, whose maximum degree is bounded from above by a constant that depends only on the dimension d , a technical challenge is that the maximum degree in an optimal PA graphs cannot be bounded by a constant in the dimension. This holds even for the simplest case of $d = 1$ and $p > 1$. We conjecture that the same holds also for $p = 1$, but proving this seems to be more difficult and not to add much.

Lemma 3.1. *For all $p > 1$, all integers $d \geq 1$, and for infinitely many n , there exists instances of n points in $[0, 1]^d$ such that the unique optimal PA graph is a tree with a maximum degree of $n - 1$.*

Proof. Let n be odd, and let $2m + 1 = n$. Consider the instance

$$X_m = \{a_{-m}, a_{-m+1}, \dots, a_0, \dots, a_{m-1}, a_m\}$$

that consists of m positive integers a_1, \dots, a_m , m negative integers $a_{-i} = -a_i$ for $1 \leq i \leq m$, and $a_0 = 0$. We assume that $a_{i+1} \gg a_i$ for all i . By scaling and shifting, we can achieve that X fits into the unit interval.

A possible solution $\text{pa} : X_m \rightarrow \mathbb{R}^+$ is assigning power a_i^p to a_i and a_{-i} for $1 \leq i \leq m$ and power a_m^p to 0. In this way, all points are connected to 0. We claim that this power assignment is the unique optimum. As $a_m = -a_{-m} \gg |a_i|$ for $|i| < m$, the dominant term in the power consumption Ψ_m is $3a_m^p$ (the power of a_m , a_{-m} , and $a_0 = 0$). Note that no other term in the total power consumption involves a_m .

We show that a_m and a_{-m} must be connected to 0 in an optimal PA graph. First, assume that a_m and a_{-m} are connected to different vertices. Then the total power consumption increases to about $4a_m^p$ because $a_{\pm m}$ is very large compared to a_i for all $|i| < m$ (we say that a_m is dominant). Second, assume that a_m and a_{-m} are connected to a_i with $i \neq 0$. Without loss of generality, we assume that $i > 0$ and, thus, $a_i > 0$. Then the total power consumption is at least $2 \cdot (a_m + a_i)^p + (a_m - a_i)^p \geq 3a_m^p + 2a_m^{p-1}a_i$. Because a_m is dominant, this is strictly more than Ψ_m because it contains the term $2a_m^{p-1}a_i$, which contains the very large a_m because $p > 1$.

From now on, we can assume that $0 = a_0$ is connected to $a_{\pm m}$. Assume that there is some point a_i that is connected to some a_j with $i, j \neq 0$. Assume without loss of generality that $i > 0$ and $|i| \geq |j|$. Assume further that i is maximal in the sense that there is no $|k| > i$ such that a_k is connected to some vertex other than 0. We set a_i 's power to a_i^p and a_j 's power to $|a_j|^p$. Then both are connected to 0 as 0 has already sufficient power to send to both. Furthermore, the PA graph is still connected: All vertices a_k with $|k| > i$ are connected to 0 by the choice of i . If some a_k with $|k| \leq i$ and $k \neq i, j$ was connected to a_i before, then it has also sufficient power to send to 0.

The power balance remains to be considered: If $j = -i$, then the energy of both a_i and a_j has been strictly decreased. Otherwise, $|j| < i$. The power of a_i was at least $(a_i - a_j)^p$ before and is now a_i^p . The power of a_j was at least $(a_i - a_j)^p$ before and is now a_j^p . Since a_i dominates all a_j for $|j| < i$, this decreases the power. \square

The unboundedness of the degree of PA graphs make the analysis of the functional PA challenging. The technical reason is that removing a vertex can cause the PA graph to fall into a non-constant number of components. The following lemma is the crucial ingredient to get over this “degree hurdle”.

Lemma 3.2. *Let $x, y \in X$, let $v \in [0, 1]^d$, and assume that x and y have power $\text{pa}(x) \geq \|x - v\|^p$ and $\text{pa}(y) \geq \|y - v\|^p$, respectively. Assume further that $\|x - v\| \leq \|y - v\|$ and that $\angle(x, v, y) \leq \alpha$ with $\alpha \leq \pi/3$. Then the following holds:*

- (a) $\text{pa}(y) \geq \|x - y\|^p$, i.e., y has sufficient power to reach x .
- (b) If x and y are not connected (i.e., $\text{pa}(x) < \|x - y\|^p$), then $\|y - v\| > \frac{\sin(2\alpha)}{\sin(\alpha)} \cdot \|x - v\|$.

Proof. Because $\alpha \leq \pi/3$, we have $\|y - v\| \geq \|y - x\|$. This implies (a).

The point x has sufficient power to reach any point within a radius of $\|x - v\|$ of itself. By (a), point y has sufficient power to send to x . Thus, if y is within a distance of $\|x - v\|$ of x , then also x can send to y and, thus, x and y are connected. We project x , y , and v into the two-dimensional subspace spanned by the vectors $x - v$ and $y - v$. This yields a situation as depicted in Figure 1. Since $\text{pa}(x) \geq \|x - v\|^p$, point x can send to all points in the light-gray region, thus in particular to all dark-gray points in the cone rooted at v . In particular, x

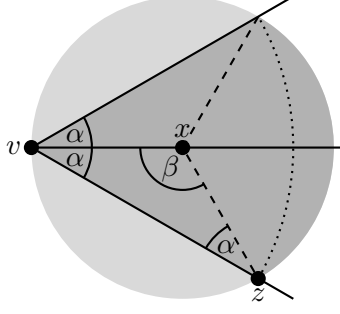


Figure 1: Point x can send to all points in the gray area as it can send to v . In particular, x can send to all points that are no further away from v than z . This includes all points to the left of the dotted line. The dotted line consists of points at a distance of $\frac{\sin(2\alpha)}{\sin(\alpha)} \cdot \|x - v\|$ of v .

can send to all points that are no further away from v than the point z . The triangle vxz is isosceles. Thus, also the angle at z is α and the angle at x is $\beta = \pi - 2\alpha$. Using the law of sines together with $\sin(\beta) = \sin(2\alpha)$ yields that $\|z - v\| = \frac{\sin(2\alpha)}{\sin(\alpha)} \cdot \|x - v\|$, which completes the proof of (b). \square

For instance, $\alpha = \pi/6$ results in a factor of $\sqrt{3} = \sin(\pi/3)/\sin(\pi/6)$. In the following, we invoke this lemma always with $\alpha = \pi/6$, but this choice is arbitrary as long as $\alpha < \pi/3$, which causes $\sin(2\alpha)/\sin(\alpha)$ to be strictly larger than 1.

3.2 Deterministic Properties

In this section, we state properties of the power assignment functional. Subadditivity (Lemma 3.3), superadditivity (Lemma 3.4), and growth bound (Lemma 3.5) are straightforward.

Lemma 3.3 (subadditivity). *PA is subadditive [25, (2.2)] for all $p > 0$ and all $d \geq 1$, i.e., for any point sets X and Y and any hyperrectangle $R \subseteq \mathbb{R}^d$ with $X, Y \subseteq R$, we have*

$$\text{PA}(X \cup Y) \leq \text{PA}(X) + \text{PA}(Y) + O((\text{diam } R)^p).$$

Proof. Let T_X and T_Y be optimal PA graphs for X and Y , respectively. We connect these graphs by an edge of length at most $\text{diam } R$. This yields a solution for $X \cup Y$, i.e., a PA graph, and the additional costs are bounded from above by the length of this edge to the power p , which is bounded by $(\text{diam } R)^p$. \square

Lemma 3.4 (superadditivity). *PA_B is superadditive for all $p \geq 1$ and $d \geq 1$ [25, (3.3)], i.e., for any X , hyperrectangle $R \subseteq \mathbb{R}^d$ with $X \subseteq R$ and partition of R into hyperrectangles R_1 and R_2 , we have*

$$\text{PA}_B^p(X, R) \geq \text{PA}_B^p(X \cap R_1, R_1) + \text{PA}_B^p(X \cap R_2, R_2).$$

Proof. Let T be an optimal boundary PA graph for (X, R) . This graph restricted to R_1 and R_2 yields boundary graphs T_1 and T_2 for $(X \cap R_1, R_1)$ and $(X \cap R_2, R_2)$, respectively. The sum of the costs of T_1 and T_2 is upper bounded by the costs of T because $p \geq 1$ (splitting an edge at the border between R_1 and R_2 results in two edges whose sum of lengths to the power p is at most the length of the original edge to the power p). \square

Lemma 3.5 (growth bound). *For any $X \subseteq [0, 1]^d$ and $0 < p$ and $d \geq 1$, we have*

$$\text{PA}_B(X) \leq \text{PA}(X) \leq O\left(\max\left\{n^{\frac{d-p}{d}}, 1\right\}\right).$$

Proof. This follows from the growth bound for the MST [25, (3.7)], because $\text{MST}(X) \leq \text{PA}(X) \leq 2 \text{MST}(X)$ for all X [10]. The inequality $\text{PA}_B(X) \leq \text{PA}(X)$ holds obviously. \square

The following lemma shows that PA is smooth, which roughly means that adding or removing a few points does not have a huge impact on the function value. Its proof requires Lemma 3.2 to deal with the fact that optimal PA graphs can have unbounded degree.

Lemma 3.6. *The power assignment functional PA is smooth for all $0 < p \leq d$ [25, (3.8)], i.e.,*

$$|\text{PA}^p(X \cup Y) - \text{PA}^p(X)| = O\left(|Y|^{\frac{d-p}{d}}\right)$$

for all point sets $X, Y \subseteq [0, 1]^d$.

Proof. One direction is straightforward: $\text{PA}(X \cup Y) - \text{PA}(X)$ is bounded by $\Psi = O(|Y|^{\frac{d-p}{d}})$, because the optimal PA graph for Y has a value of at most Ψ by Lemma 3.5. Then we can take the PA graph for Y and connect it to the tree for X with a single edge, which costs at most $O(1) \leq \Psi$ because $p \leq d$.

For the other direction, consider the optimal PA graph T for $X \cup Y$. The problem is that the degrees $\deg_T(v)$ of vertices $v \in Y$ can be unbounded (Lemma 3.1). (If the maximum degree were bounded, then we could argue in the same way as for the MST functional.) The idea is to exploit the fact that removing $v \in Y$ also frees some power. Roughly speaking, we proceed as follows: Let $v \in Y$ be a vertex of possibly large degree. We add the power of v to some vertices close to v . The graph obtained from removing v and distributing its energy has only a constant number of components.

To prove this, Lemma 3.2 is crucial. We consider cones rooted at v with the following properties:

- The cones have a small angle α , meaning that for every cone C and every $x, y \in C$, we have $\angle(x, v, y) \leq \alpha$. We choose $\alpha = \pi/6$.
- Every point in $[0, 1]^d$ is covered by some cone.
- There is a finite number of cones. (This can be achieved because d is a constant.)

Let C_1, \dots, C_m be these cones. By abusing notation, let C_i also denote all points $x \in C_i \cap (X \cup Y \setminus \{v\})$ that are adjacent to v in T . For C_i , let x_i be the point in C_i that is closest to v and adjacent to v (breaking ties arbitrarily), and let y_i be the point in C_i that is farthest from v and adjacent to v (again breaking ties arbitrarily). (For completeness, we remark that then C_i can be ignored if $C_i \cap X = \emptyset$.) Let $\ell_i = \|y_i - v\|$ be the maximum distance of any point in C_i to v , and let $\ell = \max_i \ell_i$.

We increase the power of x_i by ℓ^p/m . Since the power of v is at least ℓ^p and we have m cones, we can account for this with v 's power because we remove v . Because $\alpha = \pi/6$ and x_i is closest to v , any point in C_i is closer to x_i than to v . According to Lemma 3.2(a), every point in C_i has sufficient power to reach x_i . Thus, if x_i can reach a point $z \in C_i$, then there is an established connection between them.

From this and increasing x_i 's power to at least ℓ^p/m , there is an edge between x_i and every point $z \in C_i$ that has a distance of at most $\ell/\sqrt[p]{m}$ from v . We recall that m and p are constants.

Now let $z_1, \dots, z_k \in C_i$ be the vertices in C_i that are not connected to x_i because x_i has too little power. We assume that they are sorted by increasing distance from v . Thus, $z_k = y_i$. We can assume that no two z_j and $z_{j'}$ are in the same component after removal of v . Otherwise, we can simply ignore one of the edges $\{v, z_j\}$ and $\{v, z_{j'}\}$ without changing the components.

Since z_j and z_{j+1} were connected to v and they are not connected to each other, we can apply Lemma 3.2(b), which implies that $\|z_{j+1} - v\| \geq \sqrt{3} \cdot \|z_j - v\|$. Furthermore, $\|z_1 - v\| \geq \ell/\sqrt[p]{m}$ by assumption. Iterating this argument yields $\ell = \|z_k - v\| \geq \sqrt{3}^{k-1} \|z_1 - v\| \geq \sqrt{3}^{k-1} \cdot \ell/\sqrt[p]{m}$. This implies $k \leq \log_{\sqrt{3}}(\sqrt[p]{m}) + 1$. Thus, removing v and redistributing its energy as described causes the PA graph to fall into at most a constant number of components. Removing $|Y|$ points causes the PA graph to fall into at most $O(|Y|)$ components. These components can be connected with costs $O(|Y|^{\frac{d-p}{d}})$ by choosing one point per component and applying Lemma 3.5. \square

Lemma 3.7. PA_B is smooth for all $1 \leq p \leq d$ [25, (3.8)].

Proof. The idea is essentially identical to the proof of Lemma 3.6, and we use the same notation. Again, one direction is easy. For the other direction, note that every vertex of $G = (X, E)$, with E induced by pa is connected to at most one point at the boundary. We use the same kind of cones as for Lemma 3.6. Let $v \in G$ be a vertex that we want to remove. We ignore v 's possible connection to the boundary and proceed with the remaining connections. In this way, we obtain a forest with $O(|G|)$ components. We compute a boundary PA graph for one vertex of each component and are done because of Lemma 3.5 and in the same way as in the proof of Lemma 3.6. \square

Crucial for convergence of PA is that PA, which is subadditive, and PA_B , which is superadditive, are close to each other. Then both are close to being both subadditive and superadditive. The following lemma states that indeed PA and PA_B do not differ too much for $1 \leq p < d$.

Lemma 3.8. PA is point-wise close to PA_B for $1 \leq p < d$ [25, (3.10)], i.e.,

$$|\text{PA}^p(X) - \text{PA}_B^p(X, [0, 1]^d)| = o(n^{\frac{d-p}{d}})$$

for every set $X \subseteq [0, 1]^d$ of n points.

Proof. Let T be an optimal boundary PA graph for X . Let $Q \subseteq X$ be the set of points that have a connection to the boundary of T and let ∂Q be the corresponding points on the boundary. If we remove the connections to the boundary, we obtain a graph T' . We can assume that Q contains exactly one point per connected component of the graph T' .

We use the same dyadic decomposition as Yukich [25, proof of Lemma 3.8]. This yields that the sum of transmit powers used to connect to the boundary is bounded by the maximum of $O(n^{\frac{d-p-1}{d-1}})$ and $O(\log n)$ for $p \leq d-1$ and by a constant for $p \in (d-1, d)$. We omit the proof as it is basically identical to Yukich's proof.

Let $Q \subseteq X$ be the points connected to the boundary, and let ∂Q be the points where Q connects to the boundary. We compute a minimum-weight spanning tree Z of ∂Q . (Note that we indeed compute an MST and not a PA. This is because the MST has bounded degree and PA and MST differ by at most a factor of 2.) This MST Z has a weight of

$$O\left(\max\left\{n^{\frac{d-1-p}{d-1}}, 1\right\}\right) = o\left(n^{\frac{d-p}{d}}\right)$$

according to the growth bound for MST [25, (3.7)]. and because $d > p$. If two points $\tilde{q}, \tilde{q}' \in \partial Q$ are connected in this tree, then we connect the corresponding points $q, q' \in Q$.

The question that remains is by how much the power of the vertices in Q has to be increased in order to allow the connections as described above. If $q, q' \in Q$ are connected, then an upper bound for their power is given by the p -th power of their distances to the boundary points \tilde{q} and \tilde{q}' plus the length of the edge connecting \tilde{q} and \tilde{q}' . Applying the triangle inequality for powers of metrics twice, the energy needed for connecting q and q' is at most $4^p = O(1)$ times the sum of these distances. Since the degree of Z is bounded, every vertex in Q contributes to only a constant number of edges and, thus, only to the power consumption of a constant number of other vertices. Thus, the total additional power needed is bounded by a constant times the power of connecting Q to the boundary plus the power to use Z as a PA graph. Because of the triangle inequality for powers of metrics, the bounded degree of every vertex of ∂Q in Z , and because of the dyadic decomposition mentioned above, the increase of power is in compliance with the statement of the lemma. \square

Remark 3.9. *Lemma 3.8 is an analogue of its counterpart for MST, TSP, and matching [25, Lemma 3.7] in terms of the bounds. Namely, we obtain*

$$|\text{PA}(X) - \text{PA}_B(X)| \leq \begin{cases} O(|X|^{\frac{d-p-1}{d-1}}) & \text{if } 1 \leq p < d-1, \\ O(\log |X|) & \text{if } p = d-1 \neq 1, \\ O(1) & \text{if } d-1 < p < d \text{ or } p = d-1 = 1. \end{cases}$$

3.3 Probabilistic Properties

For $p > d$, smoothed is not guaranteed to hold, and for $p \geq d$, point-wise closeness is not guaranteed to hold. But similar properties typically hold for random point sets, namely smoothness in mean (Definition 3.14) and closeness in mean (Definition 3.16). In the following, let $X = \{U_1, \dots, U_n\}$. Recall that U_1, \dots, U_n are drawn uniformly and independently from $[0, 1]^d$.

Before proving smoothness in mean, we need a statement about the longest edge in an optimal PA graph and boundary PA graph. The bound is asymptotically equal to the bound for the longest edge in an MST [6, 11, 17].

To prove our bound for the longest edge in optimal PA graphs (Lemma 3.12), we need the following two lemmas. Lemma 3.10 is essentially equivalent to a result by Kozma et al. [11], but they do not state the probability explicitly. Lemma 3.11 is a straight-forward consequence of Lemma 3.10. Variants of both lemmas are known [6, 17, 18, 23], but, for completeness, we state and prove both lemmas in the forms that we need.

Lemma 3.10. *For every $\beta > 0$, there exists a $c_{\text{ball}} = c_{\text{ball}}(\beta, d)$ such that, with a probability of at least $1 - n^{-\beta}$, every hyperball of radius $r_{\text{ball}} = c_{\text{ball}} \cdot (\log n/n)^{1/d}$ and with center in $[0, 1]^d$ contains at least one point of X in its interior.*

Proof. We sketch the simple proof. We cover $[0, 1]^d$ with hypercubes of side length $\Omega(r_{\text{ball}})$ such that every ball of radius r_{ball} – even if its center is in a corner (for a point on the boundary, still at least a $2^{-d} = \Theta(1)$ fraction is within $[0, 1]^d$) – contains at least one box. The probability that such a box does not contain a point, which is necessary for a ball to be empty, is at most $(1 - \Omega(r_{\text{ball}})^d)^n \leq n^{-\Omega(1)}$ by independence of the points in X and the definition of r_{ball} . The rest of the proof follows by a union bound over all $O(n/\log n)$ boxes. \square

We also need the following lemma, which essentially states that if z and z' are sufficiently far away, then there is – with high probability – always a point y between z and z' in the following sense: the distance of y to z is within a predefined upper bound $2r_{\text{ball}}$, and y is closer to z' than z .

Lemma 3.11. *For every $\beta > 0$, with a probability of at least $1 - n^{-\beta}$, the following holds: For every choice of $z, z' \in [0, 1]^d$ with $\|z - z'\| \geq 2r_{\text{ball}}$, there exists a point $y \in X$ with the following properties:*

- $\|z - y\| \leq 2r_{\text{ball}}$.
- $\|z' - y\| < \|z' - z\|$.

Proof. The set of candidates for y contains a ball of radius r_{ball} , namely a ball of this radius whose center is at a distance of r_{ball} from z on the line between z and z' . This allows us to use Lemma 3.10. \square

Lemma 3.12 (longest edge). *For every constant $\beta > 0$, there exists a constant $c_{\text{edge}} = c_{\text{edge}}(\beta)$ such that, with a probability of at least $1 - n^{-\beta}$, every edge of an optimal PA graph and an optimal boundary PA graph PA_B is of length at most $r_{\text{edge}} = c_{\text{edge}} \cdot (\log n/n)^{1/d}$.*

Proof. We restrict ourselves to considering PA graphs. The proof for boundary PA graphs is almost identical.

Let T be any PA graph. Let $c_{\text{edge}} = 4k^{1/p}c_{\text{ball}}/(1 - \sqrt{3}^{-p})^{1/p}$, where k is an upper bound for the number of vertices without a pairwise connection at a distance between r and $r/\sqrt{3}$ for arbitrary r . It follows from Lemma 3.2 and its proof, that k is a constant that depends only on p and d .

Note that $c_{\text{edge}} > 2c_{\text{ball}}$. We are going to show that if T contains an edge that is longer than r_{edge} , then we can find a better PA graph with a probability of at least $1 - n^{-\beta}$, which shows that T is not optimal.

Let v be a vertex incident to the longest edge of T , and let $r_{\text{big}} > r_{\text{edge}}$ be the length of this longest edge. (The longest edge is unique with a probability of 1. The node v is not unique as the longest edge connects two points.) We decrease the power of v to $r_{\text{big}}/\sqrt{3}$. This implies that v loses contact to some points – otherwise, the power assignment was clearly not optimal.

The number c_{ball} depends on the exponent β of the lemma. Let $x_1, \dots, x_{k'}$ with $k' \leq k$ be the points that were connected to v but are in different connected components than v after decreasing v 's power. This is because the only nodes that might lose their connection to v are within a distance between $r_{\text{big}}/\sqrt{3}$ and r_{big} , and there are at most k such nodes without a pairwise connection.

Consider x_1 . Let $z_0 = v$. According to Lemma 3.11, there is a point z_1 that is closer to x_1 and at most $2r_{\text{ball}}$ away from v . Iteratively for $i = 1, 2, \dots$, we distinguish three cases until this process stops:

- (i) z_i belongs to the same component as x_j for some j (z_i is closer to x_1 than z_{i-1} , but this does not imply $j = 1$). We increase z_i 's power such that z_i is able to send to z_{i-1} . If $i > 1$, then we also increase z_{i-1} 's power accordingly.
- (ii) z_i belongs to the same component as v . Then we can apply Lemma 3.11 to z_i and x_1 and find a point z_{i+1} that is closer to x_1 than z_i and at most at a distance of $2r_{\text{ball}}$ of z_i .

- (iii) z_i is within a distance of at most $2r_{\text{ball}}$ of some x_j . In this case, we increase the energy of z_i such that z_i and x_j are connected. (The energy of x_j is sufficiently large anyhow.)

Running this process once decreases the number of connected components by one and costs at most $2(2r_{\text{ball}})^p = 2^{p+1}r_{\text{ball}}^p$ additional power. We run this process $k' \leq k$ times, thus spending at most $k2^{p+1}r_{\text{ball}}^p$ of additional power. In this way, we obtain a valid PA graph.

We have to show that the new PA graph indeed saves power. To do this, we consider the power saved by decreasing v 's energy. By decreasing v 's power, we save an amount of $r_{\text{big}}^p - (r_{\text{big}}/\sqrt{3})^p > (1 - \sqrt{3}^{-p}) \cdot r_{\text{edge}}^p$. By the choice of c_{edge} , the saved amount of energy exceeds the additional amount of $k2^{p+1}r_{\text{ball}}^p$. This contradicts the optimality of the PA graph with the edge of length $r_{\text{big}} > r_{\text{edge}}$. \square

Remark 3.13. *Since the longest edge has a length of at most r_{edge} with high probability, i.e., with a probability of $1 - n^{-\Omega(1)}$, and any ball of radius r_{edge} contains roughly $O(\log n)$ points due to Chernoff's bound [15, Chapter 4], the maximum degree of an optimum PA graph of a random point set is $O(\log n)$ with high probability – contrasting Lemma 3.1.*

Yukich gave two different notions of smoothness in mean [25, (4.13) and (4.20) & (4.21)]. We use the stronger notion, which implies the other.

Definition 3.14 (smooth in mean [25, (4.20), (4.21)]). *A Euclidean functional F is called smooth in mean if, for every constant $\beta > 0$, there exists a constant $c = c(\beta)$ such that the following holds with a probability of at least $1 - n^{-\beta}$:*

$$|F(n) - F(n \pm k)| \leq ck \cdot \left(\frac{\log n}{n}\right)^{p/d}$$

and

$$|F_B(n) - F_B(n \pm k)| = ck \cdot \left(\frac{\log n}{n}\right)^{p/d}.$$

for all $0 \leq k \leq n/2$.

Lemma 3.15. *PA_B and PA are smooth in mean for all $p > 0$ and all d .*

Proof. The bound $\text{PA}(n+k) \leq \text{PA}(n) + O(k \cdot (\frac{\log n}{n})^{\frac{p}{d}})$ follows from the fact that for all k additional vertices, with a probability of at least $1 - n^{-\beta}$ for any $\beta > 0$, there is a vertex among the first n within a distance of at most $O((\log n/n)^{1/d})$ according to Lemma 3.10 (β influences the constant hidden in the O). Thus, we can connect any of the k new vertices with costs of $O((\log n/n)^{p/d})$ to the optimal PA graph for the n nodes.

Let us now show the reverse inequality $\text{PA}(n) \leq \text{PA}(n+k) + O(k \cdot (\frac{\log n}{n})^{\frac{p}{d}})$. To do this, we show that with a probability of at least $1 - n^{-\beta}$, we have

$$\text{PA}(n) \leq \text{PA}(n+1) + O\left(\left(\frac{\log n}{n}\right)^{\frac{p}{d}}\right). \quad (1)$$

Then we iterate k times to obtain the bound we aim for.

The proof of (1) is similar to the analogous inequality in Yukich's proof [25, Lemma 4.8]. The only difference is that we first have to redistribute the power of the point U_{n+1} to its closest neighbors as in the proof of Lemma 3.6. In this way, removing U_{n+1} results in a constant number of connected components. The longest edge incident to U_{n+1} has a length

of $O((\log n/n)^{1/d})$ with a probability of at least $1 - n^{-\beta}$ for any constant $\beta > 0$. Thus, we can connect these constant number number of components with extra power of at most $O((\log n/n)^{p/d})$.

The proof of

$$|\text{PA}(n) - \text{PA}(n - k)| = O\left(k \cdot \left(\frac{\log n}{n}\right)^{\frac{p}{d}}\right)$$

and the statement

$$|\text{PA}_B(n) - \text{PA}_B(n \pm k)| = O\left(k \cdot \left(\frac{\log n}{n}\right)^{\frac{p}{d}}\right)$$

for the boundary functional are almost identical. \square

Definition 3.16 (close in mean [25, (4.11)]). *A Euclidean functional F is close in mean to its boundary functional F_B if*

$$\mathbb{E}(|F(n) - F_B(n)|) = o\left(n^{\frac{d-p}{d}}\right).$$

Lemma 3.17. *PA is close in mean to PA_B for all d and $p \geq 1$.*

Proof. It is clear that $\text{PA}_B(X) \leq \text{PA}(X)$ for all X . Thus, in what follows, we prove that $\text{PA}(X) \leq \text{PA}_B(X) + o(n^{\frac{d-p}{d}})$ holds with a probability of at least $1 - n^{-\beta}$, where β influences the constant hidden in the o . This implies closeness in mean.

With a probability of at least $1 - n^{-\beta}$, the longest edge in the graph that realizes $\text{PA}_B(X)$ has a length of $c_{\text{edge}} \cdot (\log n/n)^{1/d}$ (Lemma 3.12). Thus, with a probability of at least $1 - n^{-\beta}$ for any constant $\beta > 0$, only vertices within a distance of at most $c_{\text{edge}} \cdot (\log n/n)^{1/d}$ of the boundary are connected to the boundary. As the d -dimensional unit cube is bounded by 2^d hyperplanes, the expected number of vertices that are so close to the boundary is bounded from above by $c_{\text{edge}} n 2^d \cdot (\log n/n)^{1/d} = O((\log n)^{1/d} n^{\frac{d-1}{d}})$. With a probability of at least $1 - n^{-\beta}$ for any $\beta > 0$, this number is exceeded by no more than a constant factor.

Removing these vertices causes the boundary PA graph to fall into at most $O((\log n)^{1/d} n^{\frac{d-1}{d}})$ components. We choose one vertex of every component and start the process described in the proof of Lemma 3.12 to connect all of them. The costs per connection is bounded from above by $O((\log n/n)^{p/d})$ with a probability of $1 - n^{-\beta}$ for any constant $\beta > 0$. Thus, the total costs are bounded from above by

$$O((\log n/n)^{p/d}) \cdot O((\log n)^{1/d} n^{\frac{d-1}{d}}) = O\left((\log n)^{\frac{p-1}{d}} \cdot n^{\frac{d-1-p}{d}}\right) = o(n^{\frac{d-p}{d}})$$

with a probability of at least $1 - n^{-\beta}$ for any constant $\beta > 0$. \square

4 Convergence

4.1 Standard Convergence

Our findings of Sections 3.2 yield complete convergence of PA for $p < d$ (Theorem 4.1). Together with the probabilistic properties of Section 3.3, we obtain convergence in mean in a straightforward way for all combinations of d and p (Theorem 4.2). In Sections 4.2 and 4.3, we prove complete convergence for $p \geq d$.

Theorem 4.1. *For all d and p with $1 \leq p < d$, there exists a constant $\gamma_{\text{PA}}^{d,p}$ such that*

$$\frac{\text{PA}^p(n)}{n^{\frac{d-p}{d}}}$$

converges completely to $\gamma_{\text{PA}}^{d,p}$.

Proof. This follows from the results in Section 3.2 together with results by Yukich [25, Theorem 4.1, Corollary 6.4]. \square

Theorem 4.2. *For all $p \geq 1$ and $d \geq 1$, there exists a constant $\gamma_{\text{PA}}^{d,p}$ (equal to the constant of Theorem 4.1 for $p < d$) such that*

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}(\text{PA}^p(n))}{n^{\frac{d-p}{d}}} = \lim_{n \rightarrow \infty} \frac{\mathbb{E}(\text{PA}_B^p(n))}{n^{\frac{d-p}{d}}} = \gamma_{\text{PA}}^{d,p}.$$

Proof. This follows from the results in Sections 3.2 and 3.3 together with results by Yukich [25, Theorem 4.5]. \square

4.2 Concentration with Warnke's Inequality

McDiarmid's or Azuma-Hoeffding's inequality are powerful tools to prove concentration of measure for a function that depends on many independent random variables, all of which have only a bounded influence on the function value. If we consider smoothness in mean (see Lemma 3.15), then we have the situation that the influence of a single variable is typically very small (namely $O((\log n/n)^{p/d})$), but can be quite large in the worst case (namely $O(1)$). Unfortunately, this situation is not covered by McDiarmid's or Azuma-Hoeffding's inequality. Fortunately, Warnke [24] proved a generalization specifically for the case that the influence of single variables is typically bounded and fulfills a weaker bound in the worst case.

The following theorem is a simplified version (personal communication with Lutz Warnke) of Warnke's concentration inequality [24, Theorem 2], tailored to our needs.

Theorem 4.3 (Warnke). *Let U_1, \dots, U_n be a family of independent random variables with $U_i \in [0, 1]^d$ for each i . Suppose that there are numbers $c_{\text{good}} \leq c_{\text{bad}}$ and an event Γ such that the function $F : ([0, 1]^d)^n \rightarrow \mathbb{R}$ satisfies*

$$\max_{i \in [n]} \max_{x \in [0, 1]^d} |F(U_1, \dots, U_n) - F(U_1, \dots, U_{i-1}, x, U_{i+1}, \dots, U_n)| \leq \begin{cases} c_{\text{good}} & \text{if } \Gamma \text{ holds and} \\ c_{\text{bad}} & \text{otherwise.} \end{cases} \quad (2)$$

Then, for any $t \geq 0$ and $\gamma \in (0, 1]$ and $\eta = \gamma(c_{\text{bad}} - c_{\text{good}})$, we have

$$\mathbb{P}(|F(n) - \mathbb{E}(F(n))| \geq t) \leq 2 \exp\left(-\frac{t^2}{2n(c_{\text{good}} + \eta)^2}\right) + \frac{n}{\gamma} \cdot \mathbb{P}(\neg \Gamma). \quad (3)$$

Proof sketch. There are two differences of this simplified variant to Warnke's result [24, Theorem 2]: First, the numbers c_{good} and c_{bad} do not depend on the index i but are chosen uniformly for all indices. Second, and more importantly, the event \mathcal{B} [24, Theorem 2] is not used in Theorem 4.3. In Warnke's theorem [24, Theorem 2], the event \mathcal{B} plays only a bridging

role: it is required that $\mathbb{P}(\mathcal{B}) \leq \sum_{i=1}^n \frac{1}{\gamma_i} \cdot \mathbb{P}(\neg\Gamma)$ for some $\gamma_1, \dots, \gamma_n$ that show up in the tail bound as well. Choosing $\gamma_i = \gamma$ for all i yields $\mathbb{P}(\mathcal{B}) \leq \frac{n}{\gamma} \cdot \mathbb{P}(\neg\Gamma)$. Then

$$\mathbb{P}(\mathbf{F}(n) \geq \mathbb{E}(\mathbf{F}(n)) + t \text{ and } \neg\mathcal{B}) \leq \exp\left(-\frac{t^2}{2n(c_{\text{good}} + \eta)^2}\right)$$

yields

$$\mathbb{P}(|\mathbf{F}(n) - \mathbb{E}(\mathbf{F}(n))| \geq t) \leq 2 \exp\left(-\frac{t^2}{2n(c_{\text{good}} + \eta)^2}\right) + \frac{n}{\gamma} \cdot \mathbb{P}(\neg\Gamma)$$

by observing that a two-sided tail bound can be obtained by symmetry and adding an upper bound for the probability of \mathcal{B} to the right-hand side. \square

Next, we define *typical smoothness*, which means that, with high probability, a single point does not have a significant influence on the value of \mathbf{F} , and we apply Theorem 4.3 for typically smooth functionals \mathbf{F} . The bound of $c \cdot (\log n/n)^{p/d}$ in Definition 4.4 below for the typical influence of a single point is somewhat arbitrary, but works for PA and MST. This bound is also essentially the smallest possible, as for there can be regions of diameter $c' \cdot (\log n/n)^{1/d}$ for some small constant $c' > 0$ that contain no or only a single point. It might be possible to obtain convergence results for other functionals for weaker notions of typical smoothness.

Definition 4.4 (typically smooth). *A Euclidean functional \mathbf{F} is typically smooth if, for every $\beta > 0$, there exists a constant $c = c(\beta)$ such that*

$$\max_{x \in [0,1]^d, i \in [n]} |\mathbf{F}(U_1, \dots, U_n) - \mathbf{F}(U_1, \dots, U_{i-1}, x, U_{i+1}, \dots, U_n)| \leq c \cdot \left(\frac{\log n}{n}\right)^{p/d}$$

with a probability of at least $1 - n^{-\beta}$.

Theorem 4.5 (concentration of typically smooth functionals). *Assume that \mathbf{F} is typically smooth. Then*

$$\mathbb{P}(|\mathbf{F}(n) - \mathbb{E}(\mathbf{F}(n))| \geq t) \leq O(n^{-\beta}) + \exp\left(-\frac{t^2 n^{\frac{2p}{d}-1}}{C(\log n)^{2p/d}}\right)$$

for an arbitrarily large constant $\beta > 0$ and another constant $C > 0$ that depends on β .

Proof. We use Theorem 4.3. The event Γ is that any point can change the value only by at most $O((\log n/n)^{p/d})$. Thus, $c_{\text{good}} = O((\log n/n)^{p/d})$ and $c_{\text{bad}} = O(1)$. The probability that we do not have the event Γ is bounded by $O(n^{-\beta})$ for an arbitrarily large constant β by typical smoothness. This only influences the constant hidden in the O of the definition of c_{good} .

We choose $\gamma = O((\log n/n)^{p/d})$. In the notation of Theorem 4.3, we choose $\eta = O(\gamma)$, which is possible as $c_{\text{bad}} - c_{\text{good}} \approx c_{\text{bad}} = \Theta(1)$. Using the conclusion of Theorem 4.3 yields

$$\begin{aligned} \mathbb{P}(|\mathbf{F}(n) - \mathbb{E}(\mathbf{F}(n))| \geq t) &\leq \frac{n}{\gamma} \cdot \mathbb{P}(\neg\Gamma) + \exp\left(-\frac{t^2 n^{2p/d}}{nC(\log n)^{2p/d}}\right) \\ &\leq O(n^{-\beta}) + \exp\left(-\frac{t^2 n^{2p/d}}{nC(\log n)^{2p/d}}\right) \end{aligned}$$

for some constant $C > 0$. Here, β can be chosen arbitrarily large. \square

Choosing $t = n^{\frac{d-p}{d}} / \log n$ yields a nontrivial concentration result that suffices to prove complete convergence of typically smooth Euclidean functionals.

Corollary 4.6. *Assume that F is typically smooth. Then*

$$\mathbb{P}(|F(n) - \mathbb{E}(F(n))| > n^{\frac{d-p}{d}} / \log n) \leq O\left(n^{-\beta} + \exp\left(-\frac{n}{C(\log n)^{2+\frac{2p}{d}}}\right)\right) \quad (4)$$

for any constant β and C depending on β as in Theorem 4.5.

Proof. The proof is straightforward by exploiting that the assumption that $F(n)/n^{\frac{d-p}{d}}$ converges in mean to $\gamma_F^{d,p}$ implies $\mathbb{E}(F(n)) = \Theta(n^{\frac{d-p}{d}})$. \square

4.3 Complete Convergence for $p \geq d$

In this section, we prove that typical smoothness (Definition 4.4) suffices for complete convergence. This implies complete convergence of MST and PA by Lemma 4.8 below.

Theorem 4.7. *Assume that F is typically smooth and $F(n)/n^{\frac{d-p}{d}}$ converges in mean to $\gamma_F^{d,p}$. Then $F(n)/n^{\frac{d-p}{d}}$ converges completely to $\gamma_F^{d,p}$.*

Proof. Fix any $\varepsilon > 0$. Since

$$\lim_{n \rightarrow \infty} \mathbb{E}\left(\frac{F(n)}{n^{\frac{d-p}{d}}}\right) = \gamma_F^{d,p},$$

there exists an n_0 such that

$$\mathbb{E}\left(\frac{F(n)}{n^{\frac{d-p}{d}}}\right) \in \left[\gamma_F^{d,p} - \frac{\varepsilon}{2}, \gamma_F^{d,p} + \frac{\varepsilon}{2}\right]$$

for all $n \geq n_0$.

Furthermore, there exists an n_1 such that, for all $n \geq n_1$, the probability that $F(n)/n^{\frac{d-p}{d}}$ deviates by more than $\varepsilon/2$ from its expected value is smaller than n^{-2} for all $n \geq n_1$. To see this, we use Corollary 4.6 and observe that the right-hand side of (4) is $O(n^{-2})$ for sufficiently large β and that the event on the left-hand side is equivalent to

$$\left|\frac{F(n)}{n^{\frac{d-p}{d}}} - \frac{\mathbb{E}(F(n))}{n^{\frac{d-p}{d}}}\right| > O\left(\frac{1}{\log n}\right),$$

where $O(1/\log n) < \varepsilon/2$ for sufficiently large n_1 and $n \geq n_1$. Let $n_2 = \max\{n_0, n_1\}$. Then

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left|\frac{F(n)}{n^{\frac{d-p}{d}}}\right| > \varepsilon\right) \leq n_2 + \sum_{n=n_2+1}^{\infty} n^{-2} = n_2 + O(1) < \infty.$$

\square

Although similar in flavor, smoothness in mean does not immediately imply typical smoothness or vice versa: the latter makes only a statement about *single* points at *worst-case* positions. The former only makes a statement about adding and removing *several* points at *random* positions. However, the proofs of smoothness in mean for MST and PA do not exploit this, and we can adapt them to yield typical smoothness.

Lemma 4.8. *PA and MST are typically smooth.*

Proof. We first consider PA. Replacing a point U_k by some other (worst-case) point z can be modeled by removing U_k and adding z . We observe that, in the proof of smoothness in mean (Lemma 3.15, we did not exploit that the point added is at a random position, but the proof goes through for any single point at an arbitrary position. Also the other way around, i.e., removing z and replacing it by a random point U_k , works in the same way. Thus, PA is typically smooth.

Closely examining Yukich’s proof of smoothness in mean for MST [25, Lemma 4.8] yields the same result for MST. \square

Corollary 4.9. *For all d and p with $p \geq 1$, $\text{MST}(n)/n^{\frac{d-p}{d}}$ and $\text{PA}(n)/n^{\frac{d-p}{d}}$ converge completely to constants $\gamma_{\text{MST}}^{d,p}$ and $\gamma_{\text{PA}}^{d,p}$, respectively.*

Proof. Both MST and PA are typically smooth and converge in mean. Thus, the corollary follows from Theorem 4.7. \square

Remark 4.10. *Instead of Warnke’s method of typical bounded differences, we could also have used Kutin’s extension of McDiarmid’s inequality [12, Chapter 3]. However, this inequality yields only convergence for $p \leq 2d$, which is still an improvement over the previous complete convergence of $p < d$, but weaker than what we get with Warnke’s inequality. Furthermore, Warnke’s inequality is easier to apply and a more natural extension in the following way: intuitively, one might think that we could just take McDiarmid’s inequality and add the probability that we are not in a nice situation using a simple union bound, but, in general, this is not true [24, Section 2.2].*

5 Average-Case Approximation Ratio of the MST Heuristic

In this section, we show that the average-case approximation ratio of the MST heuristic for power assignments is strictly better than its worst-case ratio of 2. First, we prove that the average-case bound is strictly (albeit marginally) better than 2 for any combination of d and p . Second, we show a simple improved bound for the 1-dimensional case.

5.1 The General Case

The idea behind showing that the MST heuristic performs better on average than in the worst case is as follows: the weight of the PA graph obtained from the MST heuristic can not only be upper-bounded by twice the weight of an MST, but it is in fact easy to prove that it can be upper-bounded by twice the weight of the heavier half of the edges of the MST [4]. Thus, we only have to show that the lighter half of the edges of the MST contributes $\Omega(n^{\frac{d-p}{d}})$ to the value of the MST in expectation.

For simplicity, we assume that the number $n = 2m + 1$ of points is odd. The case of even n is similar but slightly more technical. We draw points $X = \{U_1, \dots, U_n\}$ as described above. Let $\text{PT}(X)$ denote the power required in the power assignment obtained from the MST. Furthermore, let H denote the m heaviest edges of the MST, and let L denote the m lightest edges of the MST. We omit the parameter X since it is clear from the context. Then we have

$$H + L = \text{MST} \leq \text{PA} \leq \text{PT} \leq 2H = 2\text{MST} - 2L \leq 2\text{MST} \quad (5)$$

since the weight of the PA graph obtained from an MST can not only be upper bounded by twice the weight of a minimum-weight spanning tree, but it is easy to show that the PA graph obtained from the MST is in fact by twice the weight of the heavier half of the edges of a minimum-weight spanning tree [4].

For distances raised to the power p , the expected value of MST is $(\gamma_{\text{MST}}^{d,p} \pm o(1)) \cdot n^{\frac{d-p}{d}}$. If we can prove that the lightest m edges of the MST are of weight $\Omega(n^{\frac{d-p}{d}})$, then it follows that the MST power assignment is strictly less than twice the optimal power assignment. \mathbf{L} is lower-bounded by the weight of the lightest m edges of the whole graph without any further constraints. Let $\mathbf{A} = \mathbf{A}(X)$ denote the weight of these m lightest edges of the whole graph. Note that both \mathbf{L} and \mathbf{A} take edge lengths to the p -power, and we have $\mathbf{A} \leq \mathbf{L}$.

Let c be a small constant to be specified later on. Let $v_{d,r} = \frac{\pi^{d/2} r^d}{\Gamma(\frac{d}{2}+1)}$ be the volume of a d -dimensional ball of radius r . For compactness, we abbreviate $c_d = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}$, thus $v_{d,r} = c_d r^d$. Note that all c_d 's are constants since d is constant.

The probability P_k that a fixed vertex v has at least k other vertices within a distance of at most $r = \ell \cdot \sqrt[d]{1/n}$ for some constant $\ell > 0$ is bounded from above by

$$P_k \leq \binom{n-1}{k} \cdot v_{d,r}^k \leq \frac{n^k (c_d r^d)^k}{k!} = \frac{n^k (c_d \ell^d n^{-1})^k}{k!} = \frac{\tilde{c}^k}{k!}$$

for another constant $\tilde{c} = \ell^d c_d$. This follows from independence and a union bound. The expected number of edges of a specific vertex that have a length of at most r is thus bounded from above by

$$\sum_{k=1}^{n-1} P_k \leq \sum_{k=1}^{n-1} \frac{\tilde{c}^k}{k!} \leq \sum_{k=1}^{\infty} \frac{\tilde{c}^k}{k!} = e^{\tilde{c}} - 1.$$

By choosing ℓ appropriately small, we can achieve that $\tilde{c} \leq 1/3$. This yields $e^{\tilde{c}} - 1 < 1/2$. By linearity of expectation, the total number of edges of length at most r in the whole graph is bounded from above by $m/2$. Thus, at least $m/2$ of the lightest m edges of the whole graph have a length of at least r . Hence, the expected value of \mathbf{A} is bounded from below by

$$\frac{m}{2} \cdot r^p = \frac{m}{2} \cdot \ell^p n^{-\frac{p}{d}} \leq \frac{\ell^p}{4} \cdot n^{\frac{d-p}{d}} = C_{\mathbf{A}}^{d,p} \cdot n^{\frac{d-p}{d}}.$$

for some constant $C_{\mathbf{A}}^{d,p} > 0$. Then the expected value of PT is bounded from above by

$$\left(2\gamma_{\text{MST}}^{d,p} - 2C_{\mathbf{A}}^{d,p} + o(1)\right) \cdot n^{\frac{d-p}{d}}$$

by (5). From this and the convergence of PA, we can conclude the following theorem.

Theorem 5.1. *For any $d \geq 1$ and any $p \geq 1$, we have*

$$\gamma_{\text{MST}}^{d,p} \leq \gamma_{\text{PA}}^{d,p} \leq 2\gamma_{\text{MST}}^{d,p} - 2C_{\mathbf{A}}^{d,p} < 2\gamma_{\text{MST}}^{d,p}$$

for some constant $C_{\mathbf{A}}^{d,p} > 0$ that depends only on d and p .

By exploiting that in particular PA converges completely, we can obtain a bound on the expected approximation ratio from the above result.

Corollary 5.2. *For any $d \geq 1$ and $p \geq 1$ and sufficiently large n , the expected approximation ratio of the MST heuristic for power assignments is bounded from above by a constant strictly smaller than 2.*

Proof. The expected approximation ratio is $\mathbb{E}(\text{PT}(n)/\text{PA}(n)) = \mathbb{E}\left(\frac{\text{PT}(n)/n^{\frac{d-p}{d}}}{\text{PA}(n)/n^{\frac{d-p}{d}}}\right)$. We know that $\text{PA}(n)/n^{\frac{d-p}{d}}$ converges completely to $\gamma_{\text{PA}}^{d,p}$. This implies that the probability that $\text{PA}(n)/n^{\frac{d-p}{d}}$ deviates by more than $\varepsilon > 0$ from $\gamma_{\text{PA}}^{d,p}$ is $o(1)$ for any $\varepsilon > 0$.

If $\text{PA}(n)/n^{\frac{d-p}{d}} \in [\gamma_{\text{PA}}^{d,p} - \varepsilon, \gamma_{\text{PA}}^{d,p} + \varepsilon]$, then the expected approximation ratio can be bounded from above by $\frac{2\gamma_{\text{MST}}^{d,p} - 2C_{\text{A}}^{d,p}}{\gamma_{\text{PA}}^{d,p} - \varepsilon}$. This is strictly smaller than 2 for a sufficiently small $\varepsilon > 0$.

Otherwise, we bound the expected approximation ratio by the worst-case ratio of 2, which contributes only $o(1)$ to its expected value. \square

Remark 5.3. *Complete convergence of the functional PT as well as smoothness and closeness in mean has been shown for the specific case $p = d$ [4]. We believe that PT converges completely for all p and d . Since then $\gamma_{\text{PT}}^{d,p} \leq 2\gamma_{\text{MST}}^{d,p} - 2C_{\text{A}}^{d,p} < 2\gamma_{\text{MST}}^{d,p}$, we would obtain a simpler proof of Corollary 5.2.*

5.2 An Improved Bound for the One-Dimensional Case

The case $d = 1$ is much simpler than the general case, because the MST is just a Hamiltonian path starting at the left-most and ending at the right-most point. Furthermore, we also know precisely what the MST heuristic does: assume that a point x_i lies between x_{i-1} and x_{i+1} . The MST heuristic assigns power $\text{PA}(x_i) = \max\{|x_i - x_{i-1}|, |x_i - x_{i+1}|\}^p$ to x_i . The example that proves that the MST heuristic is no better than a worst-case 2-approximation shows that it is bad if x_i is very close to either side and good if x_i is approximately in the middle between x_{i-1} and x_{i+1} .

In order to show an improved bound for the approximation ratio of the MST heuristic for $d = 1$, we introduce some notation. First we remark that for $X = \{U_1, \dots, U_n\}$ with high probability, there is no subinterval of length $c \log n/n$ of $[0, 1]$ that does not contain any of the n points U_1, \dots, U_n (see Lemma 3.10 for the precise statement).

We assume that no interval of length $c \log n/n$ is empty for some sufficiently large constant c for the rest of this section.

We proceed as follows: Let $x_0 = 0$, $x_{n+1} = 1$, and let $x_1 \leq \dots \leq x_n$ be the n points (sorted in increasing order) that are drawn uniformly and independently from the interval $[0, 1]$.

Now we distribute the weight of the power assignment $\text{PT}(X)$ in the power assignment obtained from the MST, and the weight of the MST as follows: For the power assignment, every point x_i (for $1 \leq i \leq n$) gets a charge of $P_i = \max\{x_i - x_{i-1}, x_{i+1} - x_i\}^p$. This is precisely the power that this point needs in the power assignment obtained from the spanning tree. For the minimum spanning tree, we divide the power of an edge (x_{i-1}, x_i) (for $1 \leq i \leq n+1$) evenly between x_{i-1} and x_i . This means that the charge of x_i is $M_i = \frac{1}{2} \cdot ((x_i - x_{i-1})^p + (x_{i+1} - x_i)^p)$.

The length of the minimum spanning tree is thus

$$\text{MST} = \underbrace{\sum_{i=1}^n M_i}_{M^*} + \underbrace{\frac{1}{2} \cdot ((x_1 - x_0)^p + (x_{n+1} - x_n)^p)}_{=M'}.$$

The total power for the power assignment obtained from this tree is

$$\text{PT} = \underbrace{\sum_{i=1}^n P_i}_{P^\star} + \underbrace{(x_1 - x_0)^p + (x_{n+1} - x_n)^p}_{=P'}.$$

Note the following: If the largest empty interval has a length of at most $c \log n/n$, then the terms P' and M' are negligible according to the following lemma. Thus, we ignore P' and M' afterwards to simplify the analysis.

Lemma 5.4. *Assume that the largest empty interval has a length of at most $c \log n/n$. Then $M' = O(M^\star \cdot \frac{(\log n)^p}{n})$ and $P' = O(P^\star \cdot \frac{(\log n)^p}{n})$.*

Proof. We have $M' \leq (c \log n/n)^p$ and $P' \leq 2(c \log n/n)^p$ because $x_1 \leq c \log n/n$ and $x_n \geq 1 - c \log n/n$ by assumption. Thus, $M', P' = O((\frac{\log n}{n})^p)$. Furthermore,

$$M^\star = \sum_{i=1}^n \frac{1}{2} \cdot ((x_i - x_{i-1})^p + (x_{i+1} - x_i)^p).$$

Since $p \geq 1$, this function becomes minimal if we place x_1, \dots, x_n equidistantly. Thus,

$$M^\star \geq \sum_{i=1}^n \left(\frac{1}{n+1} \right)^p = n \cdot \left(\frac{1}{n+1} \right)^p = \Omega(n^{1-p}).$$

With a similar calculation, we obtain $P^\star = \Omega(n^{1-p})$ and the result follows. \square

For simplicity, we assume from now on that n is even. If n is odd, the proof proceeds in exactly the same way except for some changes in the indices. In order to analyze M and P , we proceed in two steps: First, we draw all points x_1, x_3, \dots, x_{n-1} (called the *odd points*). Given the locations of these points, x_i for even i (x_i is then called an *even point*) is distributed uniformly in the interval $[x_{i-1}, x_{i+1}]$. Note that we do not really draw the odd points. Instead, we let an adversary fix these points. But the adversary is not allowed to keep an interval of length $c \log n/n$ free (because randomness would not do so either with high probability). Then the sums

$$M_{\text{even}} = \sum_{i=1}^{n/2} M_{2i}$$

and

$$P_{\text{even}} = \sum_{i=1}^{n/2} P_{2i}$$

are sums of independent random variables. (Of course M_{2i} and P_{2i} are dependent.) Now let $\ell_{2i} = x_{2i+1} - x_{2i-1}$ be the length of the interval for x_{2i} . The expected value of M_{2i} is

$$\mathbb{E}(M_{2i}) = \frac{1}{\ell_{2i}} \cdot \int_0^{\ell_{2i}} \frac{1}{2} \cdot (x^p + (\ell_{2i} - x)^p) dx = \frac{\ell_{2i}^p}{p+1}.$$

Analogously, we obtain

$$\begin{aligned}\mathbb{E}(P_{2i}) &= \frac{1}{\ell_{2i}} \cdot \int_0^{\ell_{2i}} \max\{x, \ell_{2i} - x\}^p dx \\ &= \frac{2}{\ell_{2i}} \cdot \int_0^{\ell_{2i}/2} (\ell_{2i} - x)^p dx = \left(2 - \frac{1}{2^p}\right) \cdot \frac{\ell_{2i}^p}{p+1}.\end{aligned}$$

We observe that $\mathbb{E}(P_{2i})$ is a factor $2 - 2^{-p}$ greater than $\mathbb{E}(M_{2i})$. In the same way, the expected value of P_{2i+1} is a factor of $2 - 2^{-p}$ greater than the expected value of M_{2i+1} . This is already an indicator that the approximation ratio should be $2 - 2^{-p}$.

Because M_{even} and P_{even} are sums of independent random variables, we can use Hoeffding's inequality to bound the probability that they deviate from the expected values $\mathbb{E}(M_{\text{even}})$ and $\mathbb{E}(P_{\text{even}})$.

Lemma 5.5 (Hoeffding's inequality [8]). *Let X_1, \dots, X_m be independent random variables, where X_i assumes values in the interval $[a_i, b_i]$. Let $X = \sum_{i=1}^m X_i$. Then for all $t > 0$,*

$$\mathbb{P}(X - \mathbb{E}(X) \geq t) \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^m (b_i - a_i)^2}\right).$$

By symmetry, the same bound holds for $\mathbb{P}(X - \mathbb{E}(X) \leq -t)$.

Let us start with analyzing the probability that $M_{\text{even}} < (1 - n^{-1/4}) \cdot \mathbb{E}(M_{\text{even}})$. We have $m = n/2$ in the above. Furthermore, we have $b_i = \ell_{2i}^p/2$ (obtained if $x_{2i} = x_{2i-1}$ or $x_{2i} = x_{2i+1}$) and $a_i = (\ell_{2i}/2)^p$. Thus, $(b_i - a_i)^2 = \ell_{2i}^{2p} \cdot (2^{-1} - 2^{-p})^2$. If $p > 1$ is a constant, then this is $c_p \ell_{2i}^{2p}$ for some constant c_p . For $p = 1$, it is 0. However, in this case, the length of the minimum spanning tree is exactly 1, without any randomness. Thus, for $p = 1$, we do not have to apply Hoeffding's inequality.

For $p > 1$, we obtain

$$\begin{aligned}\mathbb{P}\left(M_{\text{even}} < (1 - n^{-1/4}) \cdot \mathbb{E}(M_{\text{even}})\right) &\leq \exp\left(-\frac{2n^{-1/2} \mathbb{E}(M_{\text{even}})^2}{\sum_{i=1}^{n/2} c_p \ell_{2i}^{2p}}\right) \\ &= \exp\left(-\frac{2n^{-1/2} \left(\sum_{i=1}^{n/2} \frac{\ell_{2i}^p}{p+1}\right)^2}{\sum_{i=1}^{n/2} c_p \ell_{2i}^{2p}}\right) \\ &= \exp\left(-c' n^{-1/2} \cdot \frac{\left(\sum_{i=1}^{n/2} \ell_{2i}^p\right)^2}{\sum_{i=1}^{n/2} \ell_{2i}^{2p}}\right)\end{aligned}\tag{6}$$

with $c' = \frac{2}{(p+1)^2 c_p}$. To estimate the exponent, we use the following technical lemma.

Lemma 5.6. *Let $p \geq 1$ be a constant. Let $s_1, \dots, s_m \in [0, \beta]$ be positive numbers for some $\beta > 0$ with $\sum_{i=1}^m s_i = \gamma$ for some number γ . (We assume that $m\beta \geq \gamma$.) Then*

$$\frac{\left(\sum_{i=1}^m s_i^p\right)^2}{\sum_{i=1}^m s_i^{2p}} \geq m \cdot \left(\frac{\gamma}{m\beta}\right)^p.$$

Proof. We rewrite the numerator as

$$\sum_{i=1}^m s_i^p \sum_{j=1}^m s_j^p$$

and the denominator as

$$\sum_{i=1}^m s_i^p s_i^p.$$

Now we see that the coefficient for s_i^p in the numerator is $\sum_{j=1}^m s_j^p$, and it is $s_i^p \leq \beta^p$ in the denominator. Because of $p \geq 1$, the sum $\sum_{j=1}^m s_j^p$ is convex as a function of the s_j . Thus, it becomes minimal if all s_j are equal. Thus, the numerator is bounded from below by $m \cdot (\gamma/m)^p$. \square

With these results, we obtain the following theorem.

Theorem 5.7. *For all $p \geq 1$, we have $\gamma_{\text{MST}}^{1,p} \leq \gamma_{\text{PA}}^{1,p} \leq (2 - 2^{-p}) \cdot \gamma_{\text{MST}}^{1,p}$.*

Proof. The first inequality is immediate. For the second inequality, we apply Lemma 5.6 with $\beta = \frac{4 \log n}{n}$ and $\gamma = 1 - o(1) \geq 1/2$ (the $o(1)$ stems from the fact that we have to ignore the distance $x_1 - x_0$ and even $x_{n+1} - x_n$) and $s_i = \ell_{2i}$ and $m = n/2$ to obtain a lower bound of $\frac{n}{2} \cdot \left(\frac{1}{4 \log n}\right)^p$ for the ratio of the fraction in (6).

This yields

$$\mathbb{P} \left(M_{\text{even}} < (1 - n^{-1/4}) \cdot \mathbb{E}(M_{\text{even}}) \right) \leq \exp \left(-\Omega \left(\frac{\sqrt{n}}{(\log n)^p} \right) \right).$$

In the same way, we can show that

$$\mathbb{P} \left(P_{\text{even}} > (1 + n^{-1/4}) \cdot \mathbb{E}(P_{\text{even}}) \right) \leq \exp \left(-\Omega(n^{1/4}) \right).$$

Furthermore, the same analysis can be done for P_{odd} and M_{odd} .

Thus, both the power assignment obtained from the MST and the MST are concentrated around their means, their means are at a factor of $2 - 2^{-p}$ for large n , and the MST provides a lower bound for the optimal PA. \square

The high probability bounds for the bound of $2 - 2^{-p}$ of the approximation ratio of the power assignment obtained from the spanning tree together with the observation that in case of any “failure” event we can use the worst-case approximation ratio of 2 yields the following corollary.

Corollary 5.8. *The expected approximation ratio of the MST heuristic is at most $2 - 2^{-p} + o(1)$.*

6 Conclusions and Open Problems

We have proved complete convergence of Euclidean functionals that are *typically smooth* (Definition 4.4) for the case that the power p is larger than the dimension d . The case $p > d$ appears naturally in the case of transmission questions for wireless networks.

As examples, we have obtained complete convergence for the MST (minimum-spanning tree) and the PA (power assignment) functional. To prove this, we have used a recent concentration

of measure result by Warnke [24]. His strong concentration inequality might be of independent interest to the algorithms community. As a technical challenge, we have had to deal with the fact that the degree of an optimal power assignment graph can be unbounded.

To conclude this paper, let us mention some problems for further research:

1. Is it possible to prove complete convergence of other functionals for $p \geq d$? The most prominent one would be the traveling salesman problem (TSP). However, we are not aware that the TSP is smooth in mean,
2. Concerning the average-case approximation ratio of the MST heuristic, we only proved that the approximation ratio is smaller than 2. Only for the case $d = 1$, we provided an explicit upper bound for the approximation ratio. Is it possible to provide an improved approximation ratio as a function of d and p for general d ?
3. Can Rhee’s isoperimetric inequality [20] be adapted to work for $p \geq d$? Rhee’s inequality can be used to obtain convergence for the case that the points are not identically distributed, and has for instance been used for a smoothed analysis of Euclidean functionals [2]. (Smoothed analysis has been introduced by Spielman and Teng to explain the performance of the simplex method [21]. We refer to two surveys for an overview [14,22].)
4. Can our findings about power assignments be generalized to other settings? For instance, to get a more reliable network, we may want to have higher connectivity. Another issue would be to take into account interference of signals or noise such as the SINR or related models.

Acknowledgment

We thank Samuel Kutin, Lutz Warnke, and Joseph Yukich for fruitful discussions.

References

- [1] Ernst Althaus, Gruia Calinescu, Ion I. Mandoiu, Sushil K. Prasad, N. Tchernovski, and Alexander Zelikovsky. Power efficient range assignment for symmetric connectivity in static ad hoc wireless networks. *Wireless Networks*, 12(3):287–299, 2006.
- [2] Markus Bläser, Bodo Manthey, and B. V. Raghavendra Rao. Smoothed analysis of partitioning algorithms for Euclidean functionals. *Algorithmica*, 66(2):397–418, 2013.
- [3] Andrea E. F. Clementi, Paolo Penna, and Riccardo Silvestri. On the power assignment problem in radio networks. *Mobile Networks and Applications*, 9(2):125–140, 2004.
- [4] Maurits de Graaf, Richard J. Boucherie, Johann L. Hurink, and Jan-Kees van Ommeren. An average case analysis of the minimum spanning tree heuristic for the range assignment problem. Memorandum 11259 (revised version), Department of Applied Mathematics, University of Twente, 2013.
- [5] Stefan Funke, Sören Laue, Zvi Lotker, and Rouven Naujoks. Power assignment problems in wireless communication: Covering points by disks, reaching few receivers quickly, and energy-efficient travelling salesman tours. *Ad Hoc Networks*, 9(6):1028–1035, 2011.

- [6] Piyush Gupta and P. R. Kumar. Critical power for asymptotic connectivity in wireless networks. In William M. McEneaney, G. George Yin, and Qing Zhang, editors, *Stochastic Analysis, Control, Optimization and Applications: A Volume in Honor of W.H. Fleming*, Systems & Control: Foundations & Applications, pages 547–566. Springer, 1999.
- [7] Magnús M. Halldórsson, Stephan Holzer, Pradipta Mitra, and Roger Wattenhofer. The power of non-uniform wireless power. In *Proc. of the 24th Ann. ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1595–1606. SIAM, 2013.
- [8] Wassily Hoeffding. Probability inequalities for sums of bounded random variables. *Journal of the American Statistical Association*, 58(301):13–30, 1963.
- [9] Thomas Kesselheim. A constant-factor approximation for wireless capacity maximization with power control in the SINR model. In *Proc. of the 22nd Ann. ACM-SIAM Symp. on Discrete Algorithms (SODA)*, pages 1549–1559. SIAM, 2011.
- [10] Lefteris M. Kirousis, Evangelos Kranakis, Danny Krizanc, and Andrzej Pelc. Power consumption in packet radio networks. *Theoretical Computer Science*, 243(1-2):289–305, 2000.
- [11] Gady Kozma, Zvi Lotker, and Gideon Stupp. On the connectivity threshold for general uniform metric spaces. *Information Processing Letters*, 110(10):356–359, 2010.
- [12] Samuel Kutin. *Algorithmic and Ensemble-Based Learning*. PhD thesis, University of Chicago, Department of Computer Science, 2002.
- [13] Errol L. Lloyd, Rui Liu, Madhav V. Marathe, Ram Ramanathan, and S. S. Ravi. Algorithmic aspects of topology control problems for ad hoc networks. In *Proc. of the 3rd ACM Int. Symp. on Mobile Ad Hoc Networking and Computing (MobiHoc)*, pages 123–134. ACM, 2002.
- [14] Bodo Manthey and Heiko Röglin. Smoothed analysis: Analysis of algorithms beyond worst case. *it – Information Technology*, 53(6):280–286, 2011.
- [15] Michael Mitzenmacher and Eli Upfal. *Probability and Computing: Randomized Algorithms and Probabilistic Analysis*. Cambridge University Press, 2005.
- [16] Kaveh Pahlavan and Allen H. Levesque. *Wireless Information Networks*. Wiley, 1995.
- [17] Mathew D. Penrose. The longest edge of the random minimal spanning tree. *The Annals of Applied Probability*, 7(2):340–361, 1997.
- [18] Mathew D. Penrose. A strong law for the longest edge of the minimal spanning tree. *The Annals of Probability*, 27(1):246–260, 1999.
- [19] Theodore S. Rappaport. *Wireless Communication*. Prentice Hall, 2002.
- [20] WanSoo T. Rhee. A matching problem and subadditive Euclidean functionals. *The Annals of Applied Probability*, 3(3):794–801, 1993.
- [21] Daniel A. Spielman and Shang-Hua Teng. Smoothed analysis of algorithms: Why the simplex algorithm usually takes polynomial time. *Journal of the ACM*, 51(3):385–463, 2004.

- [22] Daniel A. Spielman and Shang-Hua Teng. Smoothed analysis: An attempt to explain the behavior of algorithms in practice. *Communications of the ACM*, 52(10):76–84, 2009.
- [23] J. Michael Steele. Probabilistic and worst case analyses of classical problems of combinatorial optimization in Euclidean space. *Mathematics of Operations Research*, 15(4):749–770, 1990.
- [24] Lutz Warnke. On the method of typical bounded differences. arXiv:1212.5796 [math.CO], 2012.
- [25] Joseph E. Yukich. *Probability Theory of Classical Euclidean Optimization Problems*, volume 1675 of *Lecture Notes in Mathematics*. Springer, 1998.